Matrices coupled in a chain. II. Spacing functions

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Abstract. For the eigenvalues of p complex hermitian $n \times n$ matrices coupled in a chain, we give a method of calculating the spacing functions. This is a generalization of the one matrix case which has been known for a long time.

1. Introduction

Let us recall here a few facts concerning the case of one matrix. For a $n \times n$ complex hermitian matrix A with matrix elements probability density $\exp[-\mathrm{tr}\, V(A)]$, the probability density of its eigenvalues $\mathbf{x} := \{x_1, x_2, ..., x_n\}$ is [1]

$$F(\mathbf{x}) \propto \exp\left[-\sum_{j=1}^{n} V(x_j)\right] \prod_{1\le i<j\le n} (x_j - x_i)^2, \tag{1.1a}$$

$$\propto \det[K(x_i, x_j)]_{i,j=1,\ldots,n}, \tag{1.1b}$$

where $V(x)$ is a real polynomial of even order, the coefficient of the highest power being positive; $K(x, y)$ is defined by

$$K(x,y) := \exp\left[-\frac{1}{2}V(x) - \frac{1}{2}V(y)\right] \sum_{i=0}^{n-1} \frac{1}{h_i} P_i(x)P_i(y), \tag{1.2}$$

$P_i(x)$ is a real polynomial of degree i and the polynomials are chosen orthogonal with the weight $\exp[-V(x)]$,

$$\int P_i(x)P_j(x)\exp[-V(x)]dx = h_i\delta_{ij}. \tag{1.3}$$

Here and in what follows, all the integrals are taken from $-\infty$ to $+\infty$, unless explicitly stated otherwise.

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The correlation function $R_k(x_1,...,x_k)$, i.e. the density of ordered sets of k eigenvalues within small intervals around x_1, ..., x_k, ignoring the other eigenvalues, is

$$R_k(x_1,...,x_k) := \frac{n!}{(n-k)!}\int F(\mathbf{x})dx_{k+1}...dx_n$$
$$= \det[K(x_i,x_j)]_{i,j=1,\dots,k}. \tag{1.4}$$

The spacing function $E(k,I)$, i.e. the probability that a chosen domain I contains exactly k eigenvalues ($0 \le k \le n$), is

$$E(k,I) := \frac{n!}{k!(n-k)!}\int F(\mathbf{x})\left[\prod_{j=1}^{k}\chi(x_j)\right]\left[\prod_{j=k+1}^{n}[1-\chi(x_j)]\right]dx_1...dx_n$$
$$= \frac{1}{k!}\left(\frac{d}{dz}\right)^k R(z,I)\bigg|_{z=-1}, \tag{1.5}$$

where $\chi(x)$ is the characteristic function of the domain I,

$$\chi(x) := \begin{cases} 1, & \text{if } x\in I,\\ 0, & \text{otherwise,}\end{cases} \tag{1.6}$$

and $R(z,I)$ is the generating function of the integrals over I of the correlation functions $R_k(x_1,...,x_k)$,

$$R(z,I) := \int F(\mathbf{x})\prod_{j=1}^{n}[1+z\chi(x_j)]\,dx_j = \sum_{k=0}^{n}\frac{\rho_k}{k!}z^k, \tag{1.7}$$

$$\rho_k = \begin{cases} 1, & k=0,\\ \int R_k(x_1,...,x_k)\prod_{j=1}^{k}\chi(x_j)dx_j, & \text{otherwise.}\end{cases} \tag{1.8}$$

The $R(z,I)$ of Eq. (1.7) can be expressed as a determinant

$$R(z,I) = \det[G_{ij}]_{i,j=0,\dots,n-1}, \tag{1.9}$$

where, using the orthogonality, Eq. (1.3), of polynomials $P_i(x)$ and splitting the constant and linear terms in z, G_{ij} reads

$$G_{ij} = \frac{1}{h_i}\int P_i(x)P_j(x)\exp[-V(x)][1+z\chi(x)]dx = \delta_{ij}+\bar{G}_{ij}. \tag{1.10}$$

Finally, $R(z,I)$ can also be written as the Fredholm determinant

$$R(z,I) = \prod_{k=1}^{n}[1+\lambda_k(z,I)] \tag{1.11}$$

of the integral equation

$$\int N(x,y)f(y)dy = \lambda f(x), \tag{1.12}$$

where remarkably the kernel $N(x,y)$ is simply $zK(x,y)\chi(y)$ with $K(x,y)$ of Eq. (1.2). The $\lambda_i(z,I)$ are the eigenvalues of the above equation and also of the matrix $[\bar{G}_{ij}]$.

These results can be extended to a chain of p complex hermitian $n\times n$ matrices. We consider the probability density for their elements

$$\mathcal{F}(A_1,\cdots,A_p) \propto \exp\left[-\mathrm{tr}\left\{\frac{1}{2}V_1(A_1)+V_2(A_2)+\cdots+V_{p-1}(A_{p-1})+\frac{1}{2}V_p(A_p)\right\}\right]$$
$$\times \exp\left[\mathrm{tr}\left\{c_1A_1A_2+c_2A_2A_3+\cdots+c_{p-1}A_{p-1}A_p\right\}\right]. \tag{1.13}$$

Here $V_j(x)$ are real polynomials of even order with positive coefficients of their highest powers and the c_j are real constants such that all the integrals which follow converge. For each j the eigenvalues of the matrix A_j are real and will be denoted by $\mathbf{x_j} := \{x_{j1}, x_{j2}, ..., x_{jn}\}$. The probability density for the eigenvalues of all the p matrices resulting from Eq. (1.13) is [2-5]

$$F(\mathbf{x_1};...;\mathbf{x_p}) = C\exp\left[-\sum_{r=1}^{n}\left\{\frac{1}{2}V_1(x_{1r})+V_2(x_{2r})+\cdots+V_{p-1}(x_{p-1r})+\frac{1}{2}V_p(x_{pr})\right\}\right]$$
$$\times\left[\prod_{1\le r<s\le n}(x_{1s}-x_{1r})(x_{ps}-x_{pr})\right]\det\left[e^{c_1x_{1r}x_{2s}}\right]\det\left[e^{c_2x_{2r}x_{3s}}\right]\cdots\det\left[e^{c_{p-1}x_{p-1r}x_{ps}}\right] \tag{1.14}$$

$$= C\left[\prod_{1\le r<s\le n}(x_{1s}-x_{1r})(x_{ps}-x_{pr})\right]\left[\prod_{k=1}^{p-1}\det\left[w_k(x_{kr},x_{k+1s})\right]_{r,s=1,...,n}\right], \tag{1.15}$$

where

$$w_k(x,y) := \exp\left[-\frac{1}{2}V_k(x)-\frac{1}{2}V_{k+1}(y)+c_kxy\right], \tag{1.16}$$

and C is a normalisation constant such that the integral of F over all the np variables x_{ir} is one.

The correlation function

$$R_{k_1,...,k_p}(x_{11},...,x_{1k_1};...;x_{p1},...,x_{pk_p})$$
$$:= \int F(\mathbf{x_1};...;\mathbf{x_p})\prod_{j=1}^{p}\left[\frac{n!}{(n-k_j)!}\prod_{r_j=k_j+1}^{n}dx_{jr_j}\right], \tag{1.17}$$

was calculated in a previous paper [6] to be an $m\times m$ determinant ($m = k_1+...+k_p$)

$$R_{k_1,...,k_p}(x_{11},...,x_{1k_1};...;x_{p1},...,x_{pk_p})$$
$$= \det\left[K_{ij}\left(x_{ir},x_{js}\right)\right]_{i,j=1,...,p;r=1,...,k_i;s=1,...,k_j}. \tag{1.18}$$

This is the density of ordered sets of k_j eigenvalues of A_j within small intervals around x_{j1}, ..., x_{jk_j} for $j = 1, 2, ..., p$.

Here we will consider the spacing function $E(k_1, I_1; ...; k_p, I_p)$, i.e. the probability that the domain I_j contains exactly k_j eigenvalues of the matrix A_j for $j = 1, ..., p$, $0 \le k_j \le n$. The domains I_j may have overlaps. As in the one matrix case one has evidently

$$E(k_1, I_1; ...; k_p, I_p) = \frac{1}{k_1!}\left(\frac{\partial}{\partial z_1}\right)^{k_1} ... \frac{1}{k_p!}\left(\frac{\partial}{\partial z_p}\right)^{k_p} R(z_1, I_1; ...; z_p, I_p)\Bigg|_{z_1=...=z_p=-1}, \tag{1.19}$$

with the generating function

$$R(z_1, I_1; ...; z_p, I_p) = \int F(\mathbf{x_1}; ...; \mathbf{x_p}) \prod_{j=1}^{p} \prod_{r=1}^{n} [1 + z_j \chi_j(x_{jr})]\, dx_{jr} \tag{1.20}$$

$$= \sum_{k_1=0}^{n} ... \sum_{k_p=0}^{n} \frac{\rho_{k_1,...,k_p}}{k_1!...k_p!} z_1^{k_1} ... z_p^{k_p}, \tag{1.21}$$

$$\rho_{k_1,...,k_p} = \begin{cases} 1, & k_1 = ... = k_p = 0, \\ \displaystyle\prod_{j=1}^{p} \left[\int_{I_j} \prod_{r=1}^{k_j} dx_{jr}\right] R_{k_1,...,k_p}(x_{11}, \cdots, x_{1k_1}; \cdots; x_{p1}, \cdots, x_{pk_p}), & \text{otherwise,} \end{cases} \tag{1.22}$$

$\chi_j(x)$ being the characteristic function of the domain I_j, Eq. (1.6).

The function $R(z_1, I_1; ...; z_p, I_p)$ will be expressed as a $n \times n$ determinant. It will also be written as a Fredholm determinant, the kernel of which will now depend on the variables z_1, ..., z_p and the domains I_1, ..., I_p in a more involved way than in the one matrix case. In particular, it does not have the remarkable form mentioned after Eq. (1.12).

2. The generating function $R(z_1, I_1; ...; z_p, I_p)$

The expression of F, Eq. (1.15), contains a product of determinants. As the product of differences

$$\Delta(\mathbf{x_1}) = \prod_{1 \le r < s \le n} (x_{1s} - x_{1r}) \tag{2.1}$$

and $\det[w_1(x_{1r}, x_{2s})]$ are completely antisymmetric and other factors in the integrand of Eq. (1.20) are completely symmetric in the variables x_{11}, ..., x_{1n}, one can replace $\det[w_1(x_{1r}, x_{2s})]$ under the integral sign in Eq. (1.20) by a single term, say the diagonal one, and multiply by $n!$. This single term is invariant under a permutation of the variables x_{1r} and simultaneously the same permutation on the variables x_{2r}. Therefore, after integration over the x_{1r}, $r = 1, ..., n$, the integrand, excluding the factor $\det[w_2(x_{2r}, x_{3s})]$, is completely

antisymmetric in the variables x_{21}, ..., x_{2n} and so one can replace the second determinant $\det[w_2(x_{2r}, x_{3s})]$ by a single term, say the diagonal one, and multiply the result by $n!$. In this way, under the integral sign one can replace successively each of the $p-1$ determinants $\det[w_k(x_{kr}, x_{k+1s})]$ by a single term multiplying the result each time by $n!$

$$R(z_1, I_1; ...; z_p, I_p) = (n!)^{p-1} C \int \Delta(\mathbf{x_1})\Delta(\mathbf{x_p}) \left[\prod_{j=1}^{p-1}\prod_{r=1}^{n} w_j(x_{jr}, x_{j+1r})\right] \times \left[\prod_{j=1}^{p}\prod_{r=1}^{n} \Big[1 + z_j\chi_j(x_{jr})\Big] dx_{jr}\right]. \tag{2.2}$$

Recall that a polynomial is called monic when the coefficient of the highest power is one. Also recall that the product of differences $\Delta(\mathbf{x})$ can be written as a determinant

$$\Delta(\mathbf{x}) = \det[x_i^{j-1}] = \det[P_{j-1}(x_i)] = \det[Q_{j-1}(x_i)], \tag{2.3}$$

where $P_j(x)$ and $Q_j(x)$ are arbitrary monic polynomials of degree j. As usual, we will choose these polynomials real and bi-orthogonal [6]

$$\int P_j(x)(w_1 * w_2 * ... * w_{p-1})(x, y) Q_k(y) dx dy = h_j \delta_{jk}, \tag{2.4}$$

with the obvious notation

$$(f * g)(x, y) = \int f(x, \xi) g(\xi, y) d\xi. \tag{2.5}$$

The normalization constant C is [6],

$$C = (n!)^{-p} \prod_{i=0}^{n-1} h_i^{-1}. \tag{2.6}$$

Now expand the determinant as a sum over $n!$ permutations $(i) := \begin{pmatrix} 0, & \cdots, & n-1 \\ i_1, & \cdots, & i_n \end{pmatrix}$, $\pi(i)$ being its sign,

$$\det[P_{s-1}(x_{1r})] = \sum_{(i)} \pi(i) P_{i_1}(x_{11}) P_{i_2}(x_{12}) ... P_{i_n}(x_{1n}). \tag{2.7}$$

Doing the same thing for $\det[Q_{s-1}(x_{pr})]$ and integrating over all the np variables x_{jr}; $j = 1, ..., p$; $r = 1, ..., n$ in Eq. (2.2), one gets

$$\begin{aligned} R(z_1, I_1; ...; z_p, I_p) &= \frac{1}{n!} \sum_{(i)} \sum_{(j)} \pi(i)\pi(j) G_{i_1 j_1} G_{i_2 j_2} ... G_{i_n j_n} \\ &= \det[G_{ij}]_{i,j=0,...,n-1}, \end{aligned} \tag{2.8}$$

where

$$G_{ij} = \frac{1}{h_i} \int P_i(x_1) \left[\prod_{k=1}^{p-1} w_k(x_k, x_{k+1}) \right] Q_j(x_p) \left[\prod_{k=1}^{p} [1 + z_k \chi_k(x_k)] dx_k \right]. \tag{2.9}$$

When all the z_k vanish, G_{ij} is equal to δ_{ij} as a consequence of the bi-orthogonality, Eq. (2.4), of the polynomials $P_i(x)$ and $Q_i(x)$. Let us define $\bar{G}_{ij}$ as follows

$$\bar{G}_{ij} := G_{ij} - \delta_{ij}, \tag{2.10}$$

so that

$$\bar{G}_{ij} = \frac{1}{h_i} \int P_i(x_1) \left[\prod_{k=1}^{p-1} w_k(x_k, x_{k+1}) \right] Q_j(x_p) \left[\prod_{k=1}^{p} [1 + z_k \chi_k(x_k)] - 1 \right] \left[\prod_{k=1}^{p} dx_k \right]. \tag{2.11}$$

Any $n \times n$ determinant is the product of its n eigenvalues and therefore one has

$$R(z_1, I_1; ...; z_p, I_p) = \prod_{k=1}^{n} \left[1 + \lambda_k(z_1, I_1; ...; z_p, I_p) \right], \tag{2.12}$$

where the $\lambda_k(z_1, I_1; ...; z_p, I_p)$ are the n roots (not necessarily distinct, either real or pairwise complex conjugates, since $\bar{G}_{ij}$ is real) of the algebraic equation in λ

$$\det[\bar{G}_{ij} - \lambda \delta_{ij}] = 0. \tag{2.13}$$

One can always write a Fredholm integral equation with a separable kernel whose eigenvalues are identical to these (cf. reference [7] for the case of $p = 2$ matrices). Indeed, for any eigenvalue λ the system of linear equations

$$\sum_{j=0}^{n-1} \bar{G}_{ij} \xi_j = \lambda \xi_i \tag{2.14}$$

has at least one solution ξ_i, $i = 0, ..., n-1$, not all zero. Multiplying both sides of the above equation by $Q_i(x)$, summing over i and using Eq. (2.11) gives the Fredholm equation

$$\int N(x, x_p) f(x_p) dx_p = \lambda f(x), \tag{2.15}$$

where

$$f(x) := \sum_{i=0}^{n-1} \xi_i Q_i(x), \tag{2.16}$$

$$N(x, x_p) := \sum_{i=0}^{n-1} \frac{1}{h_i} Q_i(x) \int P_i(x_1) \left[\prod_{k=1}^{p-1} w_k(x_k, x_{k+1}) \right] \left[\prod_{k=1}^{p} [1 + z_k \chi_k(x_k)] - 1 \right] \left[\prod_{k=1}^{p-1} dx_k \right]. \tag{2.17}$$

Hence if λ is an eigenvalue of the matrix $[\bar{G}_{ij}]$, it is also an eigenvalue of the integral equation (2.15). Conversely, since the kernel $N(x, x_p)$ is a sum of separable ones and since the polynomials $Q_i(x)$ for $i = 0, ..., n-1$ are linearly independent, if λ and $f(x)$ are, respectively an eigenvalue and an eigenfunction of this integral equation, then $f(x)$ is necessarily of the form

$$f(x) = \sum_{i=0}^{n-1} \xi_i Q_i(x), \tag{2.18}$$

and the ξ_i, $i = 0, ..., n-1$, not all zero, satisfy Eq. (2.14). Therefore λ is a root of Eq. (2.13).

When one considers the eigenvalues of a single matrix anywhere in the chain, disregarding those of the other matrices, everything works as if one is dealing with the one matrix case and formulas (1.2), (1.5), (1.7) and (1.11) are valid with minor replacements. Similarly, when one considers properties of the eigenvalues of k $(1 \le k \le p)$ matrices situated anywhere in the chain, not necessarily consecutive, everything works as if one is dealing with a chain of only k matrices; the presence of other matrices modifying only the couplings.

To say something more about the general case seems difficult.

When $V_j(x) = a_j x^2$, $j = 1, ..., p$, then the polynomials $P_j(x)$ and $Q_j(x)$ are Hermite polynomials $P_j(x) = H_j(\alpha x)$, $Q_j(x) = H_j(\beta x)$, the constants α and β depending on the parameters a_j and the couplings c_j. In this particular case the calculation can be pushed to the end (see appendix).

Appendix

For $V_j(x) = a_j x^2$, $j = 1, ..., p$, setting

$$W_{a,b,c}(x,y) := \exp\left(-\frac{1}{2}ax^2 - \frac{1}{2}by^2 + cxy\right), \tag{A.1}$$

one gets according to Eq. (2.5) the multiplication law

$$\left(W_{a,b,c} * W_{a',b',c'}\right)(x,y) = \left(\frac{2\pi}{b+a'}\right)^{1/2} W_{a'',b'',c''}(x,y), \tag{A.2}$$

where

$$a'' = a - \frac{c^2}{b+a'}, \qquad b'' = b' - \frac{c'^2}{b+a'}, \qquad c'' = \frac{cc'}{b+a'}. \tag{A.3}$$

From Eq. (1.16), $w_k(x,y) = W_{a_k, a_{k+1}, c_k}(x,y)$ and a repeated use of the above multiplication law yields

$$W(x,y) := (w_1 * w_2 * \cdots * w_{p-1})(x,y) = d \times W_{a,b,c}(x,y), \tag{A.4}$$

where a, b, c and d are constants depending on the parameters $a_1, ..., a_p$ and $c_1, ..., c_{p-1}$.

The orthogonality relation (2.4) of the polynomials $P_j(x)$ and $Q_j(x)$ takes the form

$$\int P_j(x)W(x,y)Q_k(y)dxdy = h_j\delta_{jk}, \tag{A.5}$$

namely the same relation as in the two matrix case with the weight $W(x,y)$, an exponential of a quadratic form in x and y. It follows that $P_j(x)$ and $Q_j(x)$ are Hermite polynomials of x times a constant

$$P_j(x) = H_j(\alpha x), \qquad \alpha := \left(\frac{ab-c^2}{2b}\right)^{1/2}; \tag{A.6}$$

$$Q_j(x) = H_j(\beta x), \qquad \beta := \left(\frac{ab-c^2}{2a}\right)^{1/2}; \tag{A.7}$$

$$h_j = \frac{2\pi}{(ab-c^2)^{1/2}}\left(\frac{c}{\sqrt{ab}}\right)^j 2^j j!\ d. \tag{A.8}$$

The eigenvalue density of the matrix A_1, for example, ignoring the eigenvalues of other matrices, is from Eq. (1.18)

$$\begin{aligned} R_1(x) = K_{11}(x,x) &= \sum_{j=0}^{n-1}\frac{1}{h_j}P_j(x)\int W(x,y)Q_j(y)dy \\ &= d\left(\frac{2\pi}{b}\right)^{1/2} e^{-\alpha^2x^2}\sum_{j=0}^{n-1}\frac{1}{h_j}\left(\frac{c}{\sqrt{ab}}\right)^j H_j^2(\alpha x) \\ &= \frac{\alpha}{\sqrt{\pi}}e^{-\alpha^2x^2}\sum_{j=0}^{n-1}\frac{H_j^2(\alpha x)}{2^j j!}, \end{aligned} \tag{A.9}$$

which in the large n limit is a semi-circle of radius $\sqrt{2n}/\alpha$. Thus in this particular case of coupled matrices one recovers Wigner's "semi-circle law" for the eigenvalues of a single matrix.

The kernel of the integral equation (2.15) is given by Eq. (2.17) with $P_j(x)$, $Q_j(x)$ and h_j as given above. To get further, one has to take explicitly the domains I_j into account.

References

1. See for example, M.L. Mehta, *Random Matrices*, Academic Press, San Diego, CA, U.S.A., 1991, Chapter 3. There only the case $V(x) = x^2$ is considered, but the same method applies when $V(x)$ is any real polynomial of even order.
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